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ON THE CRITICAL FILTRATION MODE WITH EVAPORATION AT THE BORDER ZONE SEPARATING FRESH WATERS FROM SALINE WATERS BELOW*

E.N. BERESLAVSKII and V.N. EMIKH

The solution of the plane problem of steady state filtration from a system of periodically distributed channels in the border zone separating fresh waters from saline waters below, constructed in /1/ with certain constraints imposed on the rate of evaporation from the free surface, is continued beyond the restrictions shown by means of appropriate transformations. The critical mode of flow in the border zone, occurring when the rate of evaporation at the border where the saline waters are drawn into the flow is increased to prescribed value, is described analytically as well as numerically with help of a digital computer, for the separate versions. The rate of evaporation is assumed, as in /1, 2/, to be proportional to the abscissa of the points of the free surface. The channels are represented by rectilinear segments of length $2l$.

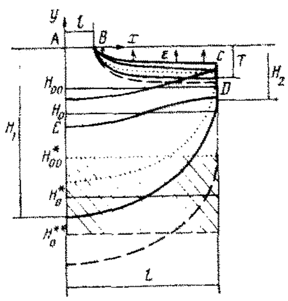

Fig.1

We model the process mathematically by a boundary value problem of determining the complex potential of the flow $\omega = \varphi + i\psi$ (φ is the filtration rate potential and ψ is the stream function), which is an analytic function of the complex coordinate $z = x + iy$, within the region $ABCDE$ (Fig.1) corresponding to a half-period of the flow. The following conditions hold along the segments of the boundary of the region z:

$$AB: y = 0,\ \varphi = 0;\ AE: x = 0,\ \psi = 0;\ CD: x = L,\ \psi = 0 \tag{1}$$
$$ED: \varphi - \rho y = \text{const},\ \psi = 0\ (\rho = \rho_2/\rho_1 - 1);\ ED: \varphi + y = 0, \psi = \varepsilon (L - x)$$

Here L is the half-distance between the middles of adjacent channels, and ρ_1 and ρ_2 are the densities of the fresh and saline waters $(\rho_1 < \rho_2)$.

The depression curve BC and the line of separation ED have to be defined. The intensity of evaporation ε was restricted in /1/ by the inequalities

$$\varepsilon < \rho,\ \varepsilon < \varepsilon_l;\ \varepsilon_l = l/(L - l) \tag{2}$$

In the problem of a fresh water lens, the first inequality in (2) follows from the constant $|W| \leqslant \rho$ along the line of separation, imposed by the premise of the immobility of saline waters on the magnitude of the filtration rate $\overline{W} = W_x + iW_y$ referred to the coefficient of filtration. In the case of the border zone, a certain excess of ε over ρ can be absorbed within the limits of the segment CD (Fig.1), without violating the restriction mentioned above and the scheme of the flow itself. It therefore follows that by virtue of the second inequality of (2) which follows from the restriction $|W| < 1$ within the contour of the channel AE, that the possibility of realizing the inequality $\varepsilon > \rho$ is governed by the relation $\varepsilon_l > \rho$.

When it holds, we have an alternative. Either increasing the parameter ε in the interval (ρ, ε_l) leads to its reaching the value $\varepsilon = \varepsilon^*$ exceeding which upsets the equilibrium between the saline waters and border zone of fresh waters, or the equilibrium will prevail over the whole range of increase in the value of parameter ε.

In order to analyse the situation, we shall transform the solution obtained earlier /1/ as it applies to the relation $\varepsilon > \rho$. As a result the basic equation becomes

$$dz/dw = -M_0[e^{\pi(\gamma+2\alpha u)}\vartheta_4(w + \alpha\varkappa, \varkappa) + e^{-\pi(\gamma+2\alpha w)}\vartheta_4(w - \alpha\varkappa, \varkappa)]\Delta^{-1} \tag{3}$$
$$M_0 > 0,\ \alpha = \frac{1}{\pi}\operatorname{arth}\sqrt{\frac{\varepsilon - \rho}{\rho(1 + \varepsilon)}},\quad \gamma = \frac{1}{\pi}\operatorname{arth}\sqrt{\frac{\varepsilon(\varepsilon - \rho)}{1 + \rho}}$$
$$\varkappa = K'/K,\ \Delta = \vartheta_4(w, \varkappa)\sqrt{a^2 - \operatorname{sn}^2(2Kw, k)},\ a = \operatorname{sn}(2Kw_A, k)$$

Here ϑ_1 and ϑ_4 are the theta functions /3/, K and K' are complete elliptic integrals of the first kind with the modulus k and $k' = \sqrt{1 - k^2}$ respectively, and sn is the elliptic Jacobi function. Fig.2 shows the domain of the parametric variable $w = u + iv$.

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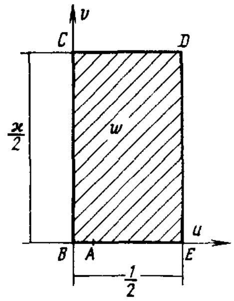

Fig.2

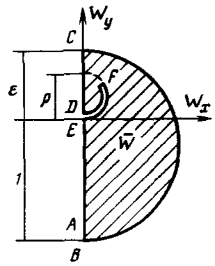

Fig.3

For the point $D\ (v = \varkappa/2)$ at the line of separation $ED\ (w = {}^1/_2 + iv, 0 \leqslant v \leqslant \varkappa/2)$, relation (3) yields

$$(dz/dv)_D = 2M\mathrm{sh}\ \pi\beta\vartheta_2\ (\alpha\varkappa,\ \varkappa)/[\vartheta_3\ (0,\ \varkappa)\sqrt{1 - a^2k^2}],\quad M = \sqrt{k}M_0, \tag{4}$$
$$\beta = \alpha + \gamma$$

From (4), taking into account the properties of the theta function ϑ_2, we conclude that

$$(dy/dx)_D = 0,\ \alpha\varkappa \neq {}^1/_2 + n,\ n = 0, 1, 2, \ldots$$

Thus the scheme of flow in the border zone with a horizontal tangent to the line of separation at the point *D*, adopted in /1/ and realized for $\varepsilon < \rho$, remains valid also for the values $\varepsilon > \rho$ satisfying the inequality $\alpha\varkappa < {}^1/_2$ and the second relation of (2).

When $\alpha\varkappa = {}^1/_2$, Eq.(3) at the line of separation *ED* will become

$$dz/dv = 2M\ \mathrm{sh}\ [\pi\ (\beta + iv/\varkappa)]\vartheta_4\ (iv, \varkappa)/ [\vartheta_2\ (iv,\ \varkappa)\sqrt{1 - a^2 + a^2k'^2\mathrm{sn}^2\ (2Kv, k')}]$$

and this will yield the relation

$$dy/dx = \mathrm{cth}\ \pi\beta\ \mathrm{tg}\ (\pi v/\varkappa) \tag{5}$$

Consequently, in the case in question the steepness of the line of separation will increase monotonically during the motion towards the point *D*, and the latter will now become a cusp: according to (5) we have at this point $dy/dx = \infty$

The velocity hodograph will also undergo corresponding transformations. In order to determine them, we shall use the following relation for the complex rate of filtration $W = W_x - iW_y = d\omega/dz$ along the line of separation:

$$W = -i\rho\ \mathrm{ch}\ \pi\beta\ V\ (v),\ 0 \leqslant v \leqslant \varkappa/2$$
$$V\ (v) = \begin{cases} \Lambda\ (0, v)\ \Lambda^{-1}\ (\beta, v), & \alpha\varkappa < {}^1/_2 \\ i\sin\ (\pi v/\varkappa)\ \mathrm{csch}\ \pi\ (\beta + iv/\varkappa), & \alpha\varkappa = {}^1/_2 \end{cases}$$
$$\Lambda\ (\beta, v) = e^{\pi(\beta+2i\alpha v)}\ \vartheta_3\ (iv + \alpha\varkappa, \varkappa) - e^{-\pi(\beta+2i\alpha v)}\ \vartheta_3\ (iv - \alpha\varkappa, \varkappa)$$

From this we find for the point *D* that $W = 0$ when $\alpha\varkappa < {}^1/_2$, $W = -i\rho$ when $\alpha\varkappa = {}^1/_2$. This means that when $\varepsilon > \rho$, the hodograph is not basically changed compared with the case $\varepsilon < \rho$, as long as $\alpha\varkappa < {}^1/_2$. The difference consists of the fact that now the segment *CD* is found on the continuation of the segment *EFD*, that is of the cut along the arc of the circle $|\overline{W} - i\rho/2| = \rho^2/4$. The tip *F* of this cut emerges on it at the point $\overline{W} = i\rho$ when $\alpha\varkappa = {}^1/_2$. According to what was said above, the point *D* also moves there from the origin of coordinates, and as a result a semicircle $|\overline{W} - i\rho/2| < \rho^2/4$ is taken out of the hodograph.

When $\alpha\varkappa = {}^1/_2$, we have at the given point $W_y = -(dp/dy)/(\rho_1 g) - 1 = \rho$, i.e. $dp/dy = -\rho_2 g$. Here the gradient of decrease in the pressure along the stream becomes equal to the gradient of hydrostatic pressure in the zone of saline waters at rest. Setting them into motion would, in this situation, require an additional infinitesimal increase in the rate of evaporation. The limit case in question reflects the critical mode of the flow in the border zone at the brink of upsetting the dynamic equilibrium of filtration of fresh waters, with saline waters below the border zone.

In order to show the premises for realizing such a mode, we shall inspect the relation connecting the initial depth H_0 of the surface of saline waters, and the maximum permissible value ε^*, in the scheme of the border zone, of the parameter ε, with the remaining three initial parameters L, l and ρ, fixed for each specific version.

As was established in /1/, for every value of $\varepsilon \in (0, \rho)$ there is a corresponding $H_0 = H_{00}\ (L, l, \rho, \varepsilon)$, at which the border zone decomposes into a chain of lenses connected to each other by the corner points (Fig.1). This value at $H_0 = H_{00}$ will therefore represent the limit value ε^* of the parameter ε. The relation $\varepsilon^*\ (H_0)$ is an increasing one, and when H_0 reaches a certain value H_{00}^*, the degeneration of the border zone will occur at the maximum value of $\varepsilon^* = \rho$ possible within the scheme of the lens. Such a limit lens and the corresponding initial depth of saline waters are shown in Fig.1 by dotted lines.

In the case of $H_0 > H_{00}^*$, the border zone will also form when $\varepsilon > \rho$. It is precisely for such depths that the critical mode of flow mentioned above should appear if the value $\varepsilon = \varepsilon^*$ inducing it falls within the range (ρ, ε_l).

The mode under discussion was studied numerically, carrying out the calculations for the relations obtained from representations (3) at $\alpha\varkappa = 1/2$. As a result we found that when the quantities L, l and ρ are fixed as agreed for every calculated value of the parameter $\varepsilon \in (0, \varepsilon_l)$, then the critical mode appears at a definite value of $H_0 = H_0^* > H_{00}^*$. The quantity H_0^* increases as ε increases. The increase in the range of possible variation in the rate of evaporation is caused, as in the case $\varepsilon < \rho$, by the increase in the depth at which the saline waters lie.

When relation $H_0^*(\varepsilon)$ is of such a character, a quantity $H_0^{**} = \lim_{\varepsilon \to \varepsilon_l} H_0^*$ representing the upper limit of the values of H_0 must exist such that the increase in the value of ε will lead, in the end, to destablization of saline waters.

When $H_0 > H_0^{**}$ on the other hand, the latter will remain static under a steady flow of fresh water in the border zone at all values of $\varepsilon \in (0, \varepsilon_l)$, within the constraint $|W| < 1$ on AE. For the given values of the depth H_0, the flow in the border zone is described in terms of the functions dz/dw and $d\omega/dw$, whose expressions when $\varepsilon < \rho$ were obtained in /1/. When $\varepsilon > \rho$, the first of these formulas is given by formulas (3).

L	$10^3\rho$	H_0	$10^3\varepsilon^*$	H_1	H_2	T	$10^4 k$	$10^4 a$
25	10	14.07	10.00	18.99	0.626	0.626	0	749
		15.00	11.21	19.90	0.832	0.698	27	694
		20.00	18.98	24.53	6.338	1.221	2078	444
		25.00	30.56	29.30	12.08	2.265	4399	198
		30.05	41.17	34.25	17.42	4.957	5603	8.7
50	10	28.48	10.00	38.35	1.597	1.597	0	252
		30.00	10.93	39.82	1.877	1.702	11	222
		35.00	14.20	44.49	6.521	2.428	681	134
		40.00	17.88	49.15	12.44	3.516	1777	51.8
		43.31	19.91	52.32	16.16	4.921	2321	9.8
50	1	27.27	1.00	37.22	0.141	0.141	0	469
		35.00	1.58	44.39	6.684	0.216	1174	394
		81.56	19.91	89.61	57.40	4.865	9003	8.1

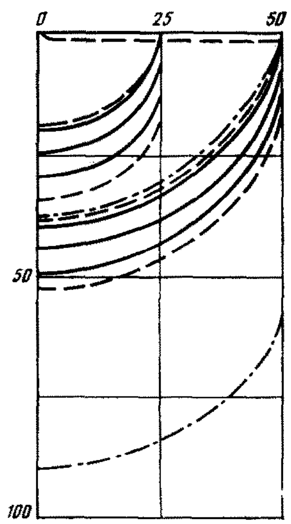

Fig.4

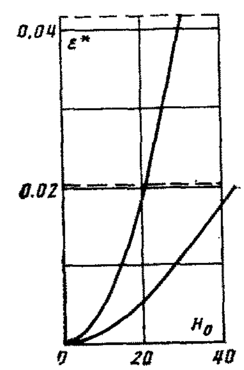

Fig.5

The table gives the results of computing the characteristic dimensions of the border zone H_1, H_2 at T, in the critical mode at $l = 1$, three combinations of the quantities L, ρ, and several values of H_0 contained within the interval (H_{00}^*, H_0^{**}) of realization of the mode (Fig.1). The left end of the interval H_{00}^* is recomputed for each version using the formulas for the lens at $\varepsilon^* = \rho$. As regards the right end, we have already mentioned that at the corresponding value of $\varepsilon^* = \varepsilon_l$ the condition $|W| < 1$ will be violated along the segment AE and the latter, together with the whole hodograph, will degenerate to the point $\overline{W} = i$ (Fig. 3). The table reflects the tendency to such degeneration: when $H_0 \to H_0^{**}, \varepsilon \to \varepsilon_l$, we have $a \to 0$ and hence the segment AE in the w plane (Fig.2) will contract to the point $w = 0$. In this connection, an increase in the value of the parameter ε used in the calculations was limited to the maximum value of $\varepsilon_0^* = \varepsilon_l - 0.0005$. The value of H_0 determined for it can be regarded as an approximation to the value H_0^{**}.

Fig.4 shows the lines of separation calculated for the combinations of initial parameters used in the table. In the series with the value $\rho = 0.01$, the extreme versions are shown by dashed lines, and in the case of $\rho = 0.001$, by the dot-dash lines. The curve for $H_0 = 35, \rho = 0.001$ was not drawn, since it is very close to the line of separation at $H_0 = 35$, $\rho = 0.01$, $L = 50$. This is confirmed by comparing the versions in question with the tabulated values of H_1 and H_2. The destabilization of saline waters of reduced density occurs when the rate of evaporation is reduced almost proportionally, and hence also the process of filtration within the border zone. In conformity with this, the maximum depth T of the free surface of the border zone is also reduced. The depression curves formally resemble each other in all the computed versions, and their positions within the ranges of the graph also differ little from each other. For this reason only one such curve is included in Fig.4 for the lens at $L = 50, \rho = 0.01$.

The relationship of ε^* described above is illustrated in Fig.5 for $\rho = 0.01$ by the upper line $(L = 25)$ and lower line $(L = 50)$. The dashed horizontal lines show the corresponding limit values of the parameter $\varepsilon = \varepsilon_l$.

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ISOPERIMETRIC INEQUALITY IN THE PROBLEM OF THE STABILITY OF A CIRCULAR RING UNDER NORMAL PRESSURE*

V.V. KOBELEV

The problem of maximizing the critical load causing a loss of stability in an elastic inextensible circular ring under hydrostatic pressure is studied. An undeformed ring has the form of a circle of unit radius, and its thickness, and hence the flexural rigidity, varies along the arc. The thickness distribution must be determined from the condition of maximum critical load causing the loss of stability, under the condition that the mass of the ring remains constant. It is shown that of all circular rings of the same mass a ring of constant thickness can bear the greatest load before losing stability.

1. Basic equations and formulation of the problem of optimization. Let us consider the conditions for the loss of stability of a circular ring acted upon by a uniformly distributed, compressive hydrostatic load. We know that under the action of hydrostatic pressure the elementary load vectors remain normal to the curved axis of the ring, and the work done by this load is equal to the product of the pressure and the difference in the areas bounded by the ring in its deformed and undeformed state. Therefore, the external load is conservative, and the phenomenon of loss of stability can be studied using static methods.

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